Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

and finally if the density were no greater than that of water the satellite, if fluid, would be unstable, the limit being actually within the surface. Since, therefore, the satellite could never have existed at its present distance in a nebular state, it must follow, if any form of the nebular hypothesis is to be accepted, that its original distance was much greater than the present. Can we find a probable cause for this ancient disturbance?

If we suppose the former period of Mars to have been very nearly one-sixth that of Jupiter the close commensurability would render the orbit of Mars more and more eccentric. The planet in perihelion would thus pass through the sun's atmosphere, or rather through the outermost equatorial zone of the solar nebula. This resisting medium would not only accelerate the motion of Mars but also in a much greater degree that of his extremely small satellite. The solar mass contracting more rapidly than the orbit of Mars would finally leave the latter moving in an excentric path without sensible resistance.

Other Secondary Systems.—For the first satellite of Jupiter the limit is 5250 miles, or 4½ times the radius of the satellite. For Mimas, the innermost satellite of Saturn, it is less than twice the radius. The rings of Saturn, in all probability, could not exist as three satellites, the limits of stability being interior to the surface.*

The effect of perturbation in the dismemberment of comets is known to all astronomers. The nucleus of the great comet of 1880, which approached within less than 100,000 miles of the sun's surface, must have had a density greater than that of granite, as well as a strong cohesive force betw'n its parts, in order to withstand the tendency to disintegration during its perihelion passage. Had the nucleus been either liquid or gaseous, or even a cluster of solid meteorites, the difference between the sun's attraction on the central and the superficial parts would have pulled the comet asunder, spreading out the fragments into somewhat different orbits, like the meteoric streams of August and November.

LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST, WATERTOWN, CONN.

IN a recent paper (ANALYST, Nov., 1880) I gave some account of what is known as Lagrange's theorem in probability, and showed how it can be applied to the determination, in the most general manner, of the law of facility of errors in one dimension, such as errors in the measured length of a line for example. The elementary errors to which any measurement is liable, give rise by their combinations to all the errors possible in the sum or the mean of several measurements. The sum of k observations is always k times their arithmetical mean, so that the sum and the mean follow the same law of facility of deviation from their most probable values. The probability of a deviation x in the sum is equal to the probability of a deviation $x \div k$ in the mean. Denoting the quadratic mean error of a single observation by r, it was noticed (p. 172) that the quadratic mean error of the sum of k observations is $r\sqrt{k}$. This is true whether k be finite or infinite, being a consequence of my theorem respecting the radius of gyration of the coefficients in the expansion of a polynomial to the k power. (ANALYST, May, 1880, p. 75.) Hence the quadratic mean error of the arithmetical mean, for any value of k, is strictly equal to

---

*It has been recently shown that Bessel's mass of the ring is much greater than the true value.

$$r\sqrt{k} \div k = r \div \sqrt{k}.$$

It will be borne in mind that r here is in strictness the square root of the mean of the squares of the deviations of all the possible values of a single observation, from their arithmetical mean, and that in forming this arithmetical mean, and the mean of the squares of the deviations, each possible value, and its squared deviation, is taken with a weight proportional to the probability of its occurrence.

If the number of observations is large, the determination of the law of facility of error reduces itself to finding the limiting form of the series of coefficients in the expansion of a polynomial to an infinitely high power. This, as I had before shown, is the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}. \tag{1}$$

We will now consider the somewhat analogous case of errors in two dimensions, or errors in the position of a point in a plane. Taking the true position of the point as an origin, and drawing through it any rect. axes, we refer to these axes the positions of the other points corresponding to the several elementary errors to which the observations are subject. Let Δx and Δy denote the units of measure in the x and y directions. If one elementary error acting alone would place the observed point at the position $x = 2\Delta x$, $y = 3\Delta y$, for example, we will say that $\lambda_{2,3}$ is the probability of the occurrence of that error; and so on for the other errors. Now write the probabilities λ as coefficients, and their sub-indices as exponents of ξ and η, in the polynomial,

$$\begin{array}{l|l} \lambda_{-m,m}\xi^{-m}\eta^{m} & \lambda_{m,m}\xi^{m}\eta^{m} \\ \hline \lambda_{-m,-m}\xi^{-m}\eta^{-m} & \lambda_{m,-m}\xi^{m}\eta^{-m} \end{array} \tag{2}$$

For convenience, only the four corner terms are written here, as in a previous article of mine (Vol. VII, p. 41). The omitted terms can be readily constructed from these, since all sub-indices, exponents and coeffic'ts which vary from term to term, are understood to vary by differences equal to unity. The greatest elementary error, positive or negative, which can occur, is supposed not to exceed m units in either the x or y directions. The polynomial, if written in full, would form a square, with $2m+1$ rows and $2m+1$ columns, containing $(2m+1)^2$ terms in all.

If any particular elementary error cannot occur, its probability λ is of course zero; and any polynomial can be put in the square form by adding terms, if necessary, with zero coefficients. By an extension of the principle of Lagrange's theorem, which a little consideration will render obvious, it appears that when k observations are made, the probability that the sum of the errors in the x direction will be $t\Delta x$, and that at the same time the sum of those in the y direction will be $v\Delta y$, is the coefficient of $\xi^t\eta^v$ in the expansion of the polynomial (2) to the k power. The same coefficient is also the probability that the mean of the x-errors will be $t \div k$ units and the mean of the y-errors $v \div k$ units. The units of measure Δx and Δy can be taken as small as we please, and the exponents in the polynomial will be regarded as whole numbers.

In k observations, the most probable total effect, in the x and y directions, of the elementary error $2\Delta x$ and $3\Delta y$ for example, will be

$$k\lambda_{2,3}(2\Delta x), \qquad k\lambda_{2,3}(3\Delta y),$$

and so on for other errors. The most probable algebraic sums of the x and y errors then will be

$$k\Delta x\left(\begin{array}{c|c} -m\lambda_{-m,m} & m\lambda_{m,m} \\ \hline -m\lambda_{-m,-m} & m\lambda_{m,-m} \end{array}\right), \qquad k\Delta y\left(\begin{array}{c|c} m\lambda_{-m,m} & m\lambda_{m,m} \\ \hline -m\lambda_{-m,-m} & -m\lambda_{m,-m} \end{array}\right), \tag{3}$$

so that these are approximately the exponents of ξ and η in that term in the expansion of (2) whose coefficient is a maximum. Hence the most probable error in the mean of the k observations, in the x and y directions respectively, is found by dividing the expressions (3) by k. It will be seen that the quotients thus obtained are the statical moments of the coefficients λ about the axes of Y and X, these coefficients being regarded as masses in a system of material points. The same quotients also represent the lever arms of the system about those axes, since the sum of all the masses λ is unity. From the principle established in my article of March '80, p. 46, it follows that the lever arms of the system of coefficients in the expansion of (2) to the k power, about the axes of X and Y respectively, are k times what they are in the first power. According to (3) therefore, the maximum coefficient in the expansion will coincide approximately with the centre of forces, or centre of gravity, of the whole expanded series of coefficients, whether k be finite or infinite. We shall hereafter get in another way a proof that this is true at the limit, when k becomes infinite.

Now to find the probability of a given error in the mean result of k observations, we must be able to find, approximately at least, the coefficient of any term of given position in the expansion of the polynomial (2) to the k power. To this end, we shall proceed to demonstrate a relation which exists between the coefficients in any square group of $(2m+1)^2$ terms in the expansion, and the co-ordinates which fix the position of the group. This is equivalent to finding a "multinomial formula" for two variables, analogous to the known formula for one variable which was employed by me in ANALYST, Sept., 1879, p. 141. That formula is demonstrated most easily by the method of indeterminate coefficients. (See for instance Bertrand's *Calcul Différentiel*, p. 330.) A similar process will be followed here, and we shall then pass to the differential equations of the surface which is the limiting form of the coefficients in the expansion of (2), in a manner somewhat analogous to that by which, in my own article just cited, the differential equation of the limiting curve was reached.

The coefficients in the expansion of a polynomial such as (2) to the k power are not altered by adding a constant number to all the exponents of ξ, or to all those of η, or to both together. If then we write

$$u = \frac{a_{0.n}\xi^0\eta^n \mid a_{n,n}\xi^n\eta^n}{a_{0,0}\xi^0\eta^0 \mid a_{n,0}\xi^n\eta^0}, \tag{4}$$

and make $n = 2m$, the whole number of terms will be the same as in (2), and if we also make all the coefficients a equal to the coefficients λ which are in the corresponding positions, the coefficients in the expansion of (4) to the k power will be the same as those in the expansion of (2). Denoting them by B, the expanded polynomial may be written

$$U = \frac{B_{0.kn}\xi^0\eta^{kn} \mid B_{kn,kn}\xi^{kn}\eta^{kn}}{B_{0,0}\xi^0\eta^0 \mid B_{kn,0}\xi^{kn}\eta^0}, \tag{5}$$

the number of terms in it being $(kn+1)^2$. We now have

$$u^k = U. \tag{6}$$

Differentiation with respect to ξ and η gives

$$ku^{k-1}\left(\frac{du}{d\xi}\right) = \frac{dU}{d\xi}, \qquad ku^{k-1}\left(\frac{du}{d\eta}\right) = \frac{dU}{d\eta}, \tag{7}$$

and dividing these by (6) and clearing of fractions, we get

$$kU\left(\frac{du}{d\xi}\right) = u\left(\frac{dU}{d\xi}\right), \qquad kU\left(\frac{du}{d\eta}\right) = u\left(\frac{dU}{d\eta}\right). \tag{8}$$

By differentiation of (4) and (5) with respect to ξ, we find

$$\frac{du}{d\xi}=\frac{1a_{1,n}\xi^0\eta^n \mid na_{n,n}\xi^{n-1}\eta^n}{1a_{1,0}\xi^0\eta^0 \mid na_{n,0}\xi^{n-1}\eta^0}, \qquad \frac{dU}{d\xi}=\frac{1B_{1,kn}\xi^0\eta^{kn} \mid knB_{kn,kn}\xi^{kn-1}\eta^{kn}}{1B_{1,0}\xi^0\eta^0 \mid knB_{kn.\,0}\xi^{kn-1}\eta^0}, \tag{9}$$

and by substitution in the first of the two equations (8),

$$\left.\begin{aligned} &k\left(\frac{B_{0,kn}\zeta^0\mu^{kn} \mid B_{kn,\,kn}\zeta^{kn}\mu^{kn}}{B_{0,\,0}\zeta^0\mu^0 \mid B_{kn,\,0}\zeta^{kn}\mu^0}\right)\left(\frac{1a_{1,\,n}\zeta^0\mu^n \mid na_{n,\,n}\zeta^{n-1}\mu^n}{1a_{1,0}\zeta^0\mu^0 \mid na_{n,0}\zeta^{n-1}\mu^0}\right)^* \\ =&\left(\frac{a_{0,\,n}\zeta^0\mu^n \mid a_{n,\,n}\zeta^n\mu^n}{a_{0,0}\zeta^0\mu^0 \mid a_{n,0}\zeta^n\mu^0}\right)\left(\frac{1B_{1,kn}\zeta^0\mu^{kn} \mid knB_{kn,kn}\zeta^{kn-1}\mu^{kn}}{1B_{1,\,0}\zeta^0\mu^0 \mid knB_{kn,\,0}\zeta^{kn-1}\mu^0}\right) \end{aligned}\right\} \tag{10}$$

Also differentiating (4) and (5) with respect to μ, and substituting in the second of the equations (8), we get a second equation analogous to (10), which I omit, to save space. The reader can easily construct it for himself. Each member of (10) contains the product of two rectangular polynomials.

If we multiply together any two such polynomials, for instance

$$\left(\frac{a_{0,\,u}\zeta^0\mu^u \mid a_{t,\,u}\zeta^t\mu^u}{a_{0,0}\zeta^0\mu^0 \mid a_{t,\,0}\zeta^t\mu^0}\right)\left(\frac{c_{0,\,w}\zeta^0\mu^w \mid c_{v,\,w}\zeta^v\mu^w}{c_{0,0}\zeta^0\mu^0 \mid c_{v,0}\zeta^v\mu^0}\right), \tag{11}$$

their product will also be a rectangular polynomial, and any term in it will have for its coefficient the sum of products of a and c, such that this sum can be expressed in the rectangular form. For example, the coefficient of $\zeta^p\mu^q$ in the product of (11) is

$$\frac{a_{0,\,q}c_{p,\,0} \mid a_{p,\,q}c_{0,\,0}}{a_{0,\,0}\,c_{p,\,q} \mid a_{p,\,0}c_{0,\,q}}. \tag{12}$$

Now forming in this way the coefficient of $\zeta^{r-1}\mu^s$ in the product in each member of (10), and equating the two to each other by the principle of indeterminate coefficients, we get

$$k\left(\frac{ra_{r,\,0}B_{0,s} \mid 1a_{1,0}B_{(r-1),s}}{ra_{r,\,s}B_{0,0} \mid 1a_{1,s}B_{(r-1),\,0}}\right)=\frac{ra_{0,\,s}B_{r,\,0} \mid 1a_{(r-1),s}B_{1,\,0}}{ra_{0.\,0}B_{r,\,s} \mid 1a_{(r-1).0}B_{1,s}}. \tag{13}$$

Likewise forming the coefficients of $\zeta^r\mu^{s-1}$ in the products in the omitted equation similar to (10), and equating them to each other, we have

$$k\left(\frac{1a_{r,\,1}B_{0,\,(s-1)} \mid 1a_{0,\,1}B_{r,(s-1)}}{s\,a_{r,\,s}B_{0,\,0} \mid s\,a_{0,\,s}B_{r,\,0}}\right)=\frac{1a_{0.\,(s-1)}B_{r,\,1} \mid 1a_{r,\,(s-1)}B_{0.1}}{s\,a_{0,0}\quad B_{r,\,s} \mid s\,a_{r,\,0}\quad B_{0,s}}. \tag{14}$$

These two equations express a relation between the coefficients a and B in the rectangular groups

$$\frac{a_{0,\,s} \,\vdots\, a_{r,\,s}}{a_{0,0} \,\vdots\, a_{r,\,0}}, \qquad \frac{B_{0,\,s} \,\vdots\, B_{r,\,s}}{B_{0,0} \,\vdots\, B_{r,\,0}}, \tag{15}$$

the one group containing coefficients in the first power of the given polynomial, while the other group contains an equal number of coefficients in its expansion to the k power. Since r and s are arbitrary, suppose each of them to be greater than n. This will not alter the value of the given polynomial, nor that of its expansion, provided we suppose the coefficients a to be equal to zero whenever either of the sub-indices of a is greater than n.

---

*For want of sorts ζ and μ are here written to represent *Xi* and *Eta*, respectively.

Omitting therefore in (13) and (14) all terms which contain such values of a as factors, we shall have remaining

$$\left.\begin{array}{l}
k\left(\begin{array}{l|l} na_{n,0}B_{(r-n),s} & 1a_{1,0}B_{(r-1),s} \\ \hline na_{n,n}B_{(r-n),(s-n)} & 1a_{1,n}\ B_{(r-1),(s-n)} \end{array}\right) = \begin{array}{l|l} ra_{0,n}B_{r,(s-n)} & (r-n)a_{n,n}B_{(r-n),(s-n)} \\ \hline ra_{0,0}B_{r,s} & (r-n)a_{n,0}B_{(r-n),s} \end{array} \\
k\left(\begin{array}{l|l} 1a_{n,1}\ B_{(r-n),(s-1)} & 1a_{0,1}\ B_{r,(s-1)} \\ \hline na_{n,n}B_{(r-n),(s-n)} & na_{0,n}\ B_{r,(s-n)} \end{array}\right) = \begin{array}{l|l} \ddagger a_{0,n}B_{r,(s-n)} & (s-n)a_{n,n}B_{(r-n),(s-n)} \\ \hline s\,a_{0,0}B_{r,s} & s\,a_{n,0}\quad B_{(r-n),s} \end{array}
\end{array}\right\}$$

$$(\ddagger = s-n). \quad (16)$$

These two equations express a relation between the coefficients a and B in the square groups

$$\begin{array}{c|c} a_{0,n} & a_{n,n} \\ \hline a_{0,0} & a_{n,0} \end{array}, \qquad \begin{array}{c|c} B_{(r-n),s} & B_{r,s} \\ \hline B_{(r-n),(s-n)} & B_{r,(s-n)} \end{array}, \qquad (17)$$

the first group containing the $(n + 1)^2$ coefficients of the given polynomial (4), while the second contains an equal number of coefficients in its expansion to the k power, these last being situated so as to form any square group, such that the highest sub-index of B in the x direction is r, and the highest in the y direction is s. The equations (16) can be advantageously modified as follows; in the first members add a column or a row, with coeffic'ts zero; and reverse the positions of the terms in the second members, as if by turning the two rectangular tables round through 180° in their own plane. We thus get

$$k\left(\begin{array}{l|l} na_{n,0}B_{(r-n),s} & 0a_{0,0}B_{r,s} \\ \hline na_{n,n}B_{(r-n),(s-n)} & 0a_{0,n}B_{r,(s-n)} \end{array}\right) = \begin{array}{l|l} (r-n)a_{n,0}B_{(r-n).s} & (r-0)a_{0\ 0}B_{r,s} \\ \hline (r-n)a_{n,n}B_{(r-n),(s-n)} & (r-0)a_{0,n}B_{r,(s-n)} \end{array}, (18)$$

while the second equation, being formed in a similar way, may be omitted here. In the second member of (18), let that portion which does not have the coefficient r be transferred to the first member, and we get

$$(k+1)\left(\begin{array}{l|l} na_{n,0}B_{(r-n),s} & 0a_{0,0}B_{r,s} \\ \hline na_{n,n}B_{(r-n),(s-n)} & 0a_{0.n}B_{r,(s-n)} \end{array}\right) = r\left(\begin{array}{l|l} a_{n,0}B_{(r-n),s} & a_{0,0}B_{r,s} \\ \hline a_{n,n}B_{(r-n),(s-n)} & a_{0,n}B_{r,(s-n)} \end{array},\right) (19)$$

the second equation being omitted as before. Since the coefficients in the polynomial (2) are represented by those in (4), and we have $n=2m$, the number of terms in each row or column is $2m + 1$ in the first power of (4), and $2km + 1$ in the expansion to the k power, so that one coefficient in the expansion will be in the middle of the square. This construction does not affect the generality of our proof of the limiting form of the expansion of a polynomial, for any given polynomial can be made to take this square form with one term in the centre, by adding terms on one or more sides, if need be, with coefficients zero. Now let us fix the position of the square group (17) of coefficients B, not by the extreme ranks r and s reckoned from the place of the first coefficient $B_{0,0}$, but by the ranks i and j of the middle term of the group, reckoned from the place of the middle coefficient in the whole expansion, in the x and y directions respectively. This gives

$$r = m(k+1)+i, \qquad\qquad s = m(k+1)+j \qquad\qquad (20)$$

Substituting these values for r and s in (19) and its companion equation, and transposing from the second members to the first ones those portions which have the coefficient $m(k+1)$, the former equation becomes

$$(k+1)\left(\begin{array}{l|l} ma_{n,0}B_{(r-n),s} & -ma_{0,0}B_{r,s} \\ \hline ma_{n,n}B_{(r-n),(s-n)} & -ma_{0,n}B_{r,(s-n)} \end{array}\right) = i\left(\begin{array}{l|l} a_{n,0}B_{(r-n)} & a_{0,0}B_{r,s} \\ \hline a_{n,n}B_{(r-n),(s-n)} & a_{0,n}B_{r,(s-n)} \end{array}\right), \quad (21)$$

and I omit the second one as before. Let us now change the notation so that instead of the coeff's a and B as in (17), we shall write λ and l, as follows,

$$\begin{array}{c:c} \lambda_{-m,m} & \lambda_{m,m} \\ \hdashline \lambda_{-m,-m} & \lambda_{m,-m} \end{array}, \qquad \begin{array}{c:c} l_{(i-m),(j+m)} & l_{(i+m),(j+m)} \\ \hdashline l_{(i-m).(j-m)} & l_{(i+m),(j-m)} \end{array}. \qquad (22)$$

Substituting these for the a and B in (21) and its companion equation, and using V as an auxiliary letter, we get the following results.

$$\left.\begin{array}{l} V = \begin{array}{l|l} \lambda_{m,-m}l_{(i-m),(j+m)} & \lambda_{-m,-m}l_{(i+m),(j+m)} \\ \hline \lambda_{m,m}\ l_{(i-m),(j-m)} & \lambda_{-m,m}\ l_{(i+m),(j-m)} \end{array} \\ \begin{array}{l|l} -m\lambda_{m,-m}l_{(i-m),(j+m)} & m\lambda_{-m,-m}l_{(i+m),(j+m)} \\ \hline -m\lambda_{m,m}\ l_{(i-m),(j-m)} & m\lambda_{-m,m}\ l_{(i+m),(j-m)} \end{array} = \left(\frac{-i}{k+1}\right)V, \\ \begin{array}{l|l} m\lambda_{m,-m}l_{(i-m).(j+m)} & m\lambda_{-m,-m}l_{(i+m),(j+m)} \\ \hline -m\lambda_{m,m}\ l_{(i-m),(j-m)} & -m\lambda_{-m,m}\ l_{(i+m),(j-m)} \end{array} = \left(\frac{-j}{k+1}\right)V. \end{array}\right\} \quad (23)$$

These two equations, like (16), express the general relation between the $(2m+1)^2$ coefficients in any polynomial of square form, and an equal number of coefficients forming a square group in any part of the expansion of the polynomial to the k power. It amounts to a "multinomial formula" for two variables. Its analogy to the formula for one variable, as given by me in ANALYST, Sept., 1879, p. 142, is apparent. The position of the sq're group of coefficients l in the expansion is fixed by the ranks i and j of its middle term reckoned from the middle coefficient of the whole expansion, so that, referring the polynomial and its expansion to the same axes, the coordinates of the middle coefficient of the group will be

$$x = i\Delta x, \qquad\qquad y = j\Delta y. \qquad\qquad (24)$$

It will be seen that the terms in (23) are made up of products of λ into l, and the particular ones to be multiplied together are readily found by rotating the λ table in (22) through 180°, in its own plane, and then applying it to the l table, and multiplying together the terms which coincide. The sum of all the products thus formed is V. And it will be seen that if these products are regarded as parallel forces acting perpendicularly to the plane of XY, and the intervals between them are taken as unity of distance, their moments about vertical and horizontal axes through the middle of the group, constitute the first members of the two final equations in (23). The relation here proved is a general one, holding true for any given polynomial, whether its coefficients λ are positive or negative.

[To be continued.]